

# COSETS OF SYLOW $p$ -SUBGROUPS AND A QUESTION OF RICHARD TAYLOR

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*For John Thompson on the occasion of his 80th birthday*

**ABSTRACT.** We prove that for any odd prime  $p$ , there exist infinitely many finite simple groups  $S$  containing a Sylow  $p$ -subgroup  $P$  of  $S$  such that some coset  $gP$  of  $P$  in  $S$  consists of elements whose order is divisible by  $p$ . This allows us to answer a question of Richard Taylor related to whether certain Galois representations are automorphic.

## 1. INTRODUCTION

In 1967, answering a question of Lowell Paige, John Thompson [6] proved the following result:

**Theorem 1.1.** *Let  $G = \mathrm{PSL}_2(53)$ . Let  $Q$  be a Sylow 2-subgroup of  $G$  (of order 4). There exists  $g \in G$  such that every element in  $gQ$  has even order.*

Thompson actually worked in  $\mathrm{SL}_2(q)$ . He wrote down a system of equations over  $\mathbb{F}_q$  with  $q \equiv \pm 3 \pmod{8}$  and observed that the existence of a coset of a Sylow 2-subgroup consisting of elements of even order amounted to finding a solution to this system of equations. He then observed that for  $q = 53$ , there was a solution. Now this can easily be checked in MAGMA [1] and indeed 53 can be replaced by 27 and also by any  $q \geq 53$  with  $q \equiv \pm 3 \pmod{8}$  (by showing that the variety defined by Thompson always has points for  $q \geq 53$ ). We have no knowledge of Paige's motivation.

Here we prove an analog for  $p$  odd. Our motivation for considering this problem was a question posed by Richard Taylor regarding what are called adequate groups. See [3, 4, 7] for more about this question and its relation to Galois representations which are automorphic.

We actually prove the following theorem:

**Theorem 1.2.** *Let  $p$  be an odd prime. Let  $D$  be the dihedral group of order  $2p$ . Let  $q$  be a prime power with  $p$  dividing  $q - 1$  and  $p^2$  not dividing  $q - 1$ . View  $D$  as a subgroup of  $G := \mathrm{PSL}_2(q)$ . If  $q$  is sufficiently large, there exists  $g \in G$  such that every element of  $gD$  has order divisible by  $p$ .*

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2010 *Mathematics Subject Classification.* Primary 20D20, 20D06; Secondary 11F80, 11R34.

Guralnick was partially supported by the NSF grant DMS-1001962 and Simons Foundation fellowship 224965.

Our method is similar to the one used by Thompson. The idea of the proof is to write down a variety parametrizing a family of solutions to the condition that every element of  $gD$  has order a multiple of  $p$ . We then prove that this variety is irreducible and defined over  $\mathbb{F}_q$  (indeed, defined over  $\mathbb{Z}[\theta]$  where  $\theta$  is a  $p$ th root of 1). We can then apply Lang-Weil [5] to conclude that there are solutions as long as  $q$  is sufficiently large.

In fact, there should be many more examples. While  $gD$  is far from being a random subset of  $G$ , if one assumes that this is the case, a straightforward computation of probabilities shows that there should be many elements  $g \in G$  with  $gD$  consisting of elements of order a multiple of  $p$ . On the other hand, one does need to be a bit careful. If  $G$  is a  $p$ -solvable group and  $P$  is a Sylow  $p$ -subgroup of  $G$ , then every coset  $gP$  contains some  $p'$ -element (let  $H$  be a  $p$ -complement in  $G$ , then  $|H \cap gP| = 1$  for all  $g \in G$ ). Our phenomenon may be related to the degree of the smallest nontrivial character of  $G$  (and the relative size of the subset being considered). See [2] for some results of this nature.

We also note that a straightforward computation using MAGMA shows that:

**Theorem 1.3.** *Let  $G = \text{PSL}_2(139)$ . Let  $Q$  be a Sylow 2-subgroup of  $G$  and let  $N := N_G(Q)$ . Then  $N \cong A_4$ . There exists a coset of  $N$  in  $G$  in which all elements have even order.*

The consequence of the above theorems is the following result in [3] which answers the question of Taylor.

**Corollary 1.4.** *Let  $p$  be prime. Let  $k$  be an algebraically closed field of characteristic  $p$ . There exists a finite group  $G$  and an absolutely irreducible faithful  $kG$ -module  $V$  such that*

- (1)  $H^1(G, k) = 0$ ;
- (2)  $p$  does not divide  $\dim V$ ;
- (3)  $H^1(G, V \otimes V^*) = 0$ ; and
- (4)  $\text{End}(V)$  is not spanned by the images of the  $p'$ -elements of  $G$ .

It is worth noting that all such examples constructed in [3] have the property that  $V$  is an induced module. It is also shown in [3] that if  $G$  is  $p$ -solvable, then (2) implies that the  $p'$ -elements do span  $\text{End}(V)$ . In [4], it was shown that if  $p \geq 2 \dim V + 2$ , then (1)-(4) hold.

## 2. FIELD EXTENSIONS

We first recall a standard consequence of Kummer theory.

**Lemma 2.1.** *Let  $p$  be a prime. Let  $E/F$  be an extension of fields with characteristic not  $p$ . Assume that  $E/F$  is Galois with elementary abelian Galois group  $G$  of order  $p^m$  and that  $F$  contains the  $p$ th roots of unity. Then  $(E^*/F^*)[p]$  has order  $p^m$ .*

**Theorem 2.2.** *Let  $D$  be a UFD with quotient field  $F$  of characteristic not dividing  $2p$  for some odd prime  $p$ . Assume that  $D$  contains  $\theta$ , a primitive  $p$ th root of 1. Let  $f_1, \dots, f_m \in D$  with the  $f_i$  square free non-units and pairwise having no common irreducible factors. Fix an algebraic closure  $L$  of  $F$ . Let  $q_i \in L$  with  $q_i^2 = f_i$ . Let  $c_i \in L$  with  $c_i^p \in D[q_i]$  and  $c_i$  not in  $D[q_i]$ . Let  $\Omega$  be the subset of  $F[c_1, \dots, c_m]$  consisting of elements of the form  $\prod_i q_i^{e_i} \prod_i c_i^{\ell_i}$  where  $e_i \in \{0, 1\}$  and  $\ell_i \in \{0, 1, \dots, p-1\}$ .*

- (1)  $[F[c_1, \dots, c_m] : F] = (2p)^m$ ;
- (2)  $D[c_1, \dots, c_m]$  is a free  $D$ -module of rank  $(2p)^m$  with basis  $\Omega$ .

*Proof.* To prove (1), we may invert all irreducible elements of  $D$  other than those dividing  $f := f_1 \dots f_m$  and so assume that  $D$  has only finitely many height 1 prime ideals. This implies that  $D$  is a semilocal PID (to see this, it suffices to reduce to the case that  $D$  is a local UFD with a single irreducible element where the result is obvious). Indeed, by inverting some further prime elements, we may assume that the  $f_i$  are distinct primes. Let  $v_i$  be the  $f_i$  valuation on  $D$ .

Let  $Q = F[q_1, \dots, q_m]$ . By Lemma 2.1  $Q$  is Galois of degree  $2^m$ . So it suffices to show that  $L := F[c_1, \dots, c_m]$  has degree  $p^m$  over  $Q$ . Let  $c = c_i$ . Since  $c^p \in Q$  and  $Q$  contains the  $p$ th roots of 1, for any field  $L'$  containing  $Q$ , either  $c \in L'$  or the minimal polynomial of  $c$  over  $L'$  is  $x^p - c^p$ .

This implies that  $L/Q$  is Galois of degree  $p^s$  for some  $s \leq m$  and moreover, the Galois group is an elementary abelian  $p$ -group.

Let  $0 \leq e_i < p$  be integers for  $i = 1, \dots, m$ . Extend each valuation  $v_i$  to a valuation  $w_i$  on  $L$ . Note that  $w_j(\prod_i c_i^{e_i}) > 0$  for some  $j$  unless  $e_i = 0$  for all  $i$ . Thus, the elements  $\prod_i c_i^{e_i}$  are all distinct modulo  $F^*$ .

Thus, the  $p$ -torsion subgroup of  $L^*/Q^*$  has order at least  $p^m$ . Now apply the previous lemma. This shows that  $[F[c_1, \dots, c_m] : F] = (2p)^m$ . Clearly,  $\Omega$  is a spanning set and therefore a basis for  $F[c_1, \dots, c_m]/F$ . Thus  $D[c_1, \dots, c_m] = D[\Omega]$  is free over  $D$  with basis  $\Omega$ .  $\square$

**Theorem 2.3.** *Let  $p$  be a prime. Let  $F$  be a field of characteristic not dividing  $2p$  containing a nontrivial  $p$ th root of unity  $\theta$ . Let  $I$  be the ideal of  $R := F[a, b, c_1, \dots, c_p]$  generated by*

$$(c_i^p \theta)^2 - (a\theta^i + b\theta^{-i})(c_i^p \theta) + 1,$$

*for  $i = 1, \dots, p$ . Then  $I$  is a prime ideal of  $R$ .*

*Proof.* Set  $d_i = \theta c_i^p$ . Thus,  $d_i^2 - (a\theta^i + b\theta^{-i})d_i + 1 \in I$ .

Note that the discriminant of this quadratic is

$$f_i := (a\theta^i + b\theta^{-i})^2 - 4 = (a\theta^i + b\theta^{-i} + 2)(a\theta^i + b\theta^{-i} - 2).$$

Note that  $f_i$  are pairwise relative prime in  $D := F[a, b]$ . Let  $\Omega$  be the set of elements of the form  $\prod_i d_i^{e_i} \prod_i c_i^{\ell_i}$  where  $e_i \in \{0, 1\}$  and  $\ell_i \in \{0, 1, \dots, p-1\}$ . Applying the previous result shows that  $\Omega$  is a basis for  $R/I$  over  $D$  since they are linearly independent in a homomorphic image.

Now  $(R/I \otimes_D Q(D))$  surjects onto the field described in the previous theorem. Comparing ranks show that  $R/I \otimes_D Q(D)$  is a field and  $R/I$  injects into this field, whence  $I$  is a prime ideal as required.  $\square$

### 3. COSETS OF SUBGROUPS

We can now prove an analog of Thompson's theorem for odd primes.

**Theorem 3.1.** *Let  $p$  be an odd prime. Let  $q$  be a prime power such that:*

- (1)  $q \equiv 1 \pmod{p}$ ;
- (2)  $p^2$  does not divide  $q - 1$ .

*Let  $G = \mathrm{PSL}_2(q)$ . Let  $P$  be a Sylow  $p$ -subgroup of  $G$  (of order  $p$ ). Let  $P \leq D$  be a dihedral group of order  $2p$  in  $G$ . If  $q$  is sufficiently large, there exists  $g \in \mathrm{PGL}_2(q)$  such that every element in  $gD$  has order divisible by  $p$ .*

*Proof.* It suffices to work in  $H = \mathrm{SL}_2(q)$ . Let  $\theta$  be a nontrivial  $p$ th root of 1  $\in \mathbb{F}_q$ . We may assume that  $P$  is generated by the diagonal matrix  $\mathrm{diag}(\theta, \theta^{-1})$  and that  $D = \langle P, x \rangle$  where

$$x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let  $g \in H$  where

$$g = \begin{pmatrix} a & -c \\ d & b \end{pmatrix}.$$

Note that an element of  $G$  has order a multiple of  $p$  if and only if its trace can be written as  $e^p \theta^j + (e^p \theta^j)^{-1}$  for some nonzero  $e \in \mathbb{F}_q$  and some  $j, 0 < j < p$  (an element of order a multiple of  $p$  can be conjugated so that it commutes with  $P$ ; i.e. it is contained in the subgroup of diagonal matrices – this subgroup is of the form  $P \times Q$  where  $Q$  has order prime to  $p$  – in particular, every element in  $Q$  is a  $p$ th power).

Consider the following  $2p$  equations:

$$\begin{aligned} a\theta^i + b\theta^{-i} &= (c_i^p \theta) + (c_i^p \theta)^{-1} \\ c\theta^i + d\theta^{-i} &= (d_i^p \theta) + (d_i^p \theta)^{-1} \end{aligned}$$

for  $i = 1, \dots, p$ . Then

$$g = \begin{pmatrix} a & -c \\ d & b \end{pmatrix}$$

has the property every element of  $gD$  has order a multiple of  $p$ .

By the previous result, these equations define a variety which is a direct product of two 2-dimensional irreducible varieties defined over any field containing the  $p$ th roots of 1. Thus, by Lang-Weil [5], for  $q$  sufficiently large, there are solutions (with  $ab + cd \neq 0$ ).  $\square$

Since  $\mathrm{PGL}_2(q) \leq \mathrm{PSL}_2(q^2)$ , replacing  $q$  by  $q^2$  (which still satisfies the various hypotheses), and so we can in fact take  $g \in \mathrm{PSL}_2(q)$  if we wish. Alternatively, we can take our two varieties which we will denote  $X(a, b)$  and  $X(c, d)$ . Let  $Y$  be the subvariety of  $X(a, b) \times X(c, d) \times \mathbb{A}^1$  defined by

$ab + cd = w^2$  (where  $w$  is the parameter on  $\mathbb{A}^1$ ). It is easy to see that (since  $X(a, b)$  and  $X(c, d)$  are irreducible), so is  $Y$  and so Lang-Weil applies to  $Y$  as well. Thus, we can choose our  $g$  above to have square determinant, whence reducing modulo the center of  $\mathrm{GL}_2(q)$ , the image of  $g$  is in  $\mathrm{PSL}_2(q)$ .

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